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DETERMINANTAL EQUATIONS IN STRUCTURAL MECHANICS

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The following mathematical problem arises frequently in connection with physical problems: Given the elements of a determinant $\Delta(x)$ as functions of a variable x, find real values of x for which $\Delta(x)$ vanishes. Two important examples that come to mind immediately are the determination of natural frequencies of a continuous structure, such as a beam, plate or shell, and the search for eigenvalues of a constant, real, symmetric matrix (where the eigenvalues play the role of x). This Note deals with a simple method for finding x, *a method that frees one from having to expand* Δ *in literal form*, which becomes an important consideration when Δ is large.

Let $y(\tau)$ denote a function of a variable τ, let k be any constant, and require that

$$y(0) = k \tag{1}$$

and

$$\Delta[y(\tau)] = (1-\tau)\Delta(k). \tag{2}$$

Then $y(1)$ satisfies $\Delta[y(1)] = 0$, which means that $y(1)$ is precisely one of the values of x being sought. To find $y(1)$, proceed as follows: Differentiate eqn (2) with respect to τ, solve the resulting equation for dy/dt, obtaining

$$\frac{dy}{d\tau} = -\frac{\Delta(k)}{\dfrac{d\Delta}{dy}} \tag{3}$$

and integrate this equation numerically from $\tau = 0$ to $\tau = 1$, using eqn (1) as an initial condition, and assigning to k a value in the vicinity of an expected value of x.

The characteristic equation for a cantilever beam of length L, flexural rigidity EI, and mass per unit of length ρ, which can be written

$$\Delta(x) \triangleq \begin{vmatrix} \sin x + \sinh x & \cos x + \cosh x \\ \cos x + \cosh x & -\sin x + \sinh x \end{vmatrix} = 0 \tag{4}$$

furnishes an illustrative example. Replacing x with y in eqn (4), and differentiating with respect to y, one arrives at

$$\frac{d\Delta}{dy} = \begin{vmatrix} \cos y + \cosh y & -\sin y + \sinh y \\ \cos y + \cosh y & -\sin y + \sinh y \end{vmatrix} + \begin{vmatrix} \sin y + \sinh y & \cos y + \cosh y \\ -\sin y + \sinh y & -\cos y + \cosh y \end{vmatrix} \tag{5}$$

so that, since the first determinant on the right-hand side is equal to zero, substitution from eqns (4) and (5) into eqn (3) yields

$$\frac{dy}{d\tau} = \frac{-\begin{vmatrix} \sin k + \sinh k & \cos k + \cosh k \\ \cos k + \cosh k & -\sin k + \sinh k \end{vmatrix}}{\begin{vmatrix} \sin y + \sinh y & \cos y + \cosh y \\ -\sin y + \sinh y & -\cos y + \cosh y \end{vmatrix}}. \tag{6}$$

Table 1.

Line	k	x	$\lvert\Delta(x)\rvert$
1	1.00000000	1.87437055	6×10^{-3}
2	1.87437055	1.87584140	6×10^{-3}
3	1.87584140	1.87510652	2×10^{-5}
4	1.87510652	1.87510284	1×10^{-5}
5	1.87510284	1.87510406	$< 10^{-6}$
6	4.00000000	4.69360687	5×10^{-2}
7	4.69360687	4.69461382	5×10^{-2}
8	4.69461382	4.69409282	2×10^{-4}
9	4.69409282	4.69409026	9×10^{-5}
10	4.69409026	4.69409112	$< 10^{-6}$
11	7.00000000	4.70582236	1
12	4.70582236	4.68123089	1
13	4.68123089	4.69405020	4×10^{-3}
14	4.69405020	4.69411222	2×10^{-3}
15	4.69411222	4.69409124	9×10^{-6}
16	4.69409124	4.69409111	$< 10^{-6}$
17	11.00000000	10.9955517	6×10^{-1}
18	10.9955517	10.9955348	3×10^{-1}
19	10.9955348	10.9955391	$< 10^{-6}$

Numerical integration of this equation, from $\tau = 0$ to $\tau = 1$, with $y(0) = k$, leads to the values recorded in Table 1.

The values of k listed in lines 1, 6, 11 and 17 were chosen arbitrarily, whereas each of the remaining k's is simply equal to the x on the preceding line, a choice that can be automated easily and that leads to good results rather quickly, as is apparent from the associated rapid decreases in $|\Delta(x)|$. The fact that many choices of k can lead to the same value of x (see lines 6 and 11 for k and lines 10 and 16 for the corresponding values of x) means that one need not possess great prescience to make a suitable choice for k. Moreover, in order to find a particular root of $\Delta(x) = 0$, one does not need to use the method more than once. The accuracy of the result one obtains from a single application of the method depends both on the proximity of k to the root in question and on the quality of the integration algorithm being employed. (The results in Table 1 were generated with a rather crude integrator on a desktop computer.)

The ideas underlying the method here set forth are the same as those employed in [1] to solve sets of nonlinear equations. They have been employed also by Shippy [2] to find eigenvalues and eigenvectors.

REFERENCES

1. T. R. Kane, Real solutions of sets of nonlinear equations. *AIAA J.* 4(10), 1880–1881 (1966).
2. D. J. Shippy, Matrix eigenvalues by numerical integration. *AIAA J.* 7(3), 531 (1969).